Rate-Distortion Performance in Coding Bandlimited Sources by Sampling and Dithered Quantization

Ram Zamir and Meir Feder, *Senior Member, IEEE*

***Abstract*— The rate-distortion characteristics of a scheme for encoding continuous-time band limited stationary sources, with a prescribed band, is considered. In this coding procedure the input is sampled at Nyquist's rate or faster, the samples undergo dithered uniform or lattice quantization, using subtractive dither, and the quantizer output is entropy-coded. The rate-distortion performance, and the tradeoff between the sampling rate and the quantization accuracy is investigated, utilizing the observation that the coding scheme is equivalent to an additive noise channel. It is shown that the mean-square error of the scheme is fixed as long as the product of the sampling period and the quantizer second moment is kept constant, while for a fixed distortion the coding rate generally increases when the sampling rate exceeds the Nyquist rate. Finally, as the lattice quantizer dimension becomes large, the equivalent additive noise channel of the scheme tends to be white Gaussian, and both the rate and the distortion performance become invariant to the sampling rate.**

***Index Terms*—Sampling, uniform/lattice quantization, dithered quantization, universal coding, bandlimited signals.**

I. Introduction

NYQUIST'S well-known sampling theorem states that a bandlimited signal can be faithfully represented via its samples taken at a rate twice its bandwidth. The samples may have continuously many values. In practice, the samples are quantized, leading to a distorted representation of the original signal. The rate-distortion characteristics of this digitization scheme may be analyzed via classical quantization theory, summarized, e.g., in [21]. Furthermore, the rate-distortion function of the bandlimited source, representing the optimal performance in compressing this source, is given by the rate-distortion function of the discrete-time process, sampled at Nyquist's rate (see, e.g., [1 p. 137]).

In theory, then, there is no need to sample the bandlimited process at a rate higher than Nyquist's rate. However, when practical quantization is examined instead of the theoretically optimal rate-distortion function, increasing the sampling rate may be advantageous. It seems that by increasing the sampling rate we may reduce the required quantization resolution and still achieve comparable rate-distortion characteristics in compressing the original signal. The practical advantages of using a smaller number of quantization levels, even a single-bit quantization accuracy, at high sampling rate, are indicated by the recently popular sigma–delta techniques [8].

As a matter of fact, while infinite resolution sampling at Nyquist's rate provided one extreme condition for perfect reconstruction, it was shown more recently (see, e.g., [15]) that under certain conditions bandlimited signals may be faithfully reconstructed by the location of their zero crossing, level crossing, or the location of their intersection with some functions. For this reconstruction the zero-crossing location must be provided with an infinite accuracy, whose specification requires an infinite number of bits. Specification at any finite precision (corresponding to a finite information rate) leads, of course, to a distorted reconstruction.

The two extreme cases that provide perfect reconstruction with an infinite amount of information seem to be understood. However, in cases where distortion is allowed, the behavior is not clear, and it is interesting to analyze the tradeoff between the sampling rate and the quantization accuracy at various values for the distortion and information rate. With this motivation in mind, we provide in this paper an explicit rate-distortion analysis of a sampling and quantization scheme for encoding continuous-time continuous-value stationary bandlimited signals. In this analysis we examine the sampling rate/quantization accuracy tradeoffs. Sampling and quantization schemes have been considered and analyzed before, e.g., in [11], [20], and elsewhere. However, unlike previous results on this subject, we are able to come up with explicit rate-distortion expressions, by considering Entropy-Coded Dithered Quantization (ECDQ), and relying on our previous results in [22] where the rate-distortion performance of ECDQ for vector sources has been analyzed.

Our analysis shows that at a fixed mean-square error, there is an extra cost in coding rate as the sampling rate increases above the Nyquist rate. In scalar quantization, for example, when we sample at twice the Nyquist rate, we may use a quantizer (or an A/D) whose number of levels is reduced by a factor of $\sqrt{2}$, and still get the same distortion, but the total coding rate will increase by approximately 0.47 bit per each original Nyquist sample. The same effect happens for lattice quantizers, although the additional coding rate becomes smaller as the lattice dimension grows.

In this paper we consider coding of continuous-time random processes. The definitions and theorems associated with information functions of such entities sometimes require complicated concepts and definitions that were originally introduced by Kolmogorov [12], and later on extended and made rigorous by Pinsker [16] and others. We try to state the main ideas of

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The authors are with the Department of Electrical Engineering–Systems, Faculty of Engineering, Tel-Aviv University, Tel-Aviv, 69978, Israel.

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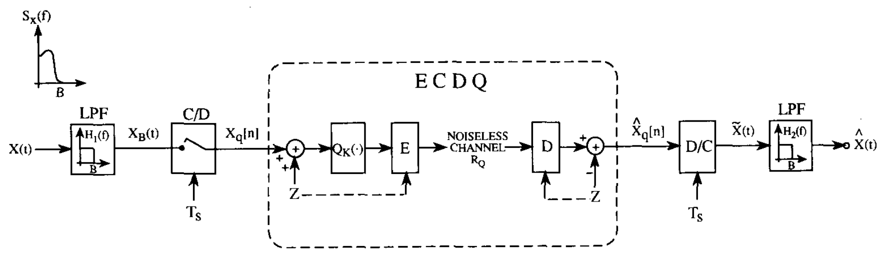
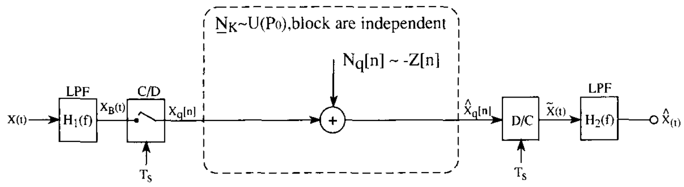


(a)



(b)

Fig. 1. (a) The sampling quantization, and coding scheme. (b) Its equivalent additive noise channel.

this paper in an intuitive manner; however, in our analysis of the coding rate and the effects of the sampling rate, in Sections IV and V, we must use the appropriate Pinsker's definition of the mutual information and the various definitions of the rate-distortion function, which in turn might have complicated the exposition. We provide some background and explanations along the paper and the Appendix, but the reader may need to consult an additional basic reference on information functions of random processes, such as [9].

II. Definitions and Scheme Description

Let $x = \{x(t), -\infty < t < \infty\}$ be a sample function of a mean-square (M.S.)-sense bandlimited source X. It is assumed that the source is stationary with a finite power σ_x^2, and that its power spectrum function is limited to the frequency band $0 \le f \le B$, i.e.,

$$S_x(f) = 0, \qquad \text{for all } f > B \tag{1}$$

where, since the power spectrum is symmetric in f, we consider in this paper only nonnegative frequencies. Besides (1) no additional knowledge on $S_x(f)$ is assumed. Throughout the paper we use capital letters, e.g., $X(t)$, $\boldsymbol{X}$, X, to denote the random variable, vector, on process, and use, e.g., $x(t)$, $\boldsymbol{x}$, x to denote their sample values.

In the proposed scheme for coding $x(t)$ the signal is passed through an "anti-aliasing filter"

$$H_1(f) = \begin{cases} 1, & \text{for } f \le B \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

and sampled at a rate $F_s = 1/T_s \ge 2B$ samples per second where $2B$ is the Nyquist rate. The filter H_1 forces *every* source realization to be bandlimited, and its necessity will become clear later. The sampled signal, $x_q = \{x_q[n], -\infty < n < \infty\}$, is transmitted via a noiseless channel, at a rate R_Q, using an ECDQ procedure with a *white* lattice quantizer (the definition of a white quantizer is provided below). The reconstructed discrete-time signal at the output of the ECDQ is denoted $\hat{x}_q$. Finally, a continuous-time distorted signal, denoted $\hat{x}(t)$, is reconstructed using an ideal digital-to-analog (D/A) converter and a bandpass filter $H_2(f) = H_1(f)$. The coding scheme is illustrated in Fig. 1(a).

The main component of our scheme is the Entropy-Coded Dithered Quantizer (ECDQ), which is a dithered lattice quantizer, with a subtractive dither, followed by lossless coding. The ECDQ has been analyzed in [25] and [22], but for completeness, we provide here its definition and some of its properties that will be needed throughout the paper. For this, we first recall definition of a lattice quantizer. A K-dimensional lattice quantizer $Q_K = \{\mathcal{L}, \mathcal{P}\}$ is defined by a set of code points $\mathcal{L} = \{l_i\}$, which form a K-dimensional lattice, and an associated partition $\mathcal{P} = \{P_i\}$ of $\mathcal{R}^K$, such that

$$P_i = l_i + P^0 = \{\boldsymbol{x}: \boldsymbol{x} - l_i \in P_0\}, \quad \boldsymbol{x} \in \mathcal{R}^K. \tag{3}$$

The quantizer maps a source vector $\boldsymbol{x}_K \in P_i$ into its associated lattice point l_i, i.e., $Q_K(\boldsymbol{x}_K) = l_i$ for $\boldsymbol{x} \in P_i$. When the mapping is into the nearest lattice point, we get the commonly used *Voronoi* partition (see [3]).

In the sequel we use the notation $Q_K(\boldsymbol{x})$, where $\boldsymbol{x} \in \mathcal{R}^n$ and K divides n, to denote a vector in $\mathcal{R}^n$ which is a concatenation of n/K successive lattice points associated with the n/K blocks of size K of $\boldsymbol{x}$. Similarly $Q_K(x)$, where x is a sequence, denotes the sequence of lattice points associated with the K-blocks of x.

A basic structure figure of the lattice quantizer, which is particularly useful when the square error distortion measure is considered, is the quantizer's normalized second moment, see [4]

$$G_K = \frac{1}{K}\frac{\int_{P_0} \|\boldsymbol{x}\|^2\, d\boldsymbol{x}}{V^{1+2/K}}, \qquad \boldsymbol{x}, \hat{\boldsymbol{x}} \in \mathcal{R}^K \tag{4}$$

where the polytope P_0 is the quantizer basic cell, and $V = \int_{P_0} d\boldsymbol{x}$ is its volume. The variance per dimension of the vector

$\boldsymbol{Z}_K$, which is uniformly distributed over P_0, is $\epsilon = G_K \cdot V^{2/K}$ (e.g., $\epsilon = \Delta^2/12$ for scalar uniform quantizer where Δ is the quantizer step). Note that unlike the scalar case where the uniform lattice is the only possible lattice, there are many possible K-dimensional lattice quantizers; thus it is desired to choose at each dimension the quantizer with the minimal G_K. It is shown in [24] that this optimal lattice quantizer has the property that the vector $\boldsymbol{Z}_K \sim \mathcal{U}(P_0)$ is *white*, i.e.

$$E\{\boldsymbol{Z}_k \boldsymbol{Z}_K^t\} = \epsilon \cdot I \tag{5}$$

where I is the identity matrix. In general, Q_K is defined to be a *white lattice quantizer* if it satisfies (5).

Subtractive dithered quantization (see [18] and [19]) is achieved by adding the random variable $\boldsymbol{Z}_K$ to every K-block of source samples before quantization, and subtracting it at reconstruction. The dither samples are drawn independently for every new K-block, and are assumed to be available to the decoder (e.g., the dither comes from a pseudo-random number generator). Thus the decoder represents a source vector $\boldsymbol{x}_K \in \mathcal{R}^K$ as

$$\hat{\boldsymbol{x}}_K = Q_K(\boldsymbol{x}_K + \boldsymbol{z}_K) - \boldsymbol{z}_K. \tag{6}$$

In ECDQ, the output of the quantizer is losslessly encoded ("entropy-coded") conditioned on the dither signal. Thus the number of bits required for ECDQ of a vector $\boldsymbol{X}_q \in \mathcal{R}^n$, where K divides n, is given by the conditional entropy of the quantizer output $H(Q_K(\boldsymbol{X}_q + \boldsymbol{Z})|\boldsymbol{Z})$, where $\boldsymbol{Z}$ is a concatenation of n/K independent replica of $\boldsymbol{Z}_K$, and where we have also neglected the possible redundancy of the lossless encoding–decoding operation.

We return now to the proposed system for encoding continuous-time signals. The coding rate of the entire system is the rate of the ECDQ block defined above, whose input is x_q and its output is $\hat{x}_q$. Following the discussion above, the asymptotic coding rate of the system is

$$\begin{aligned} R_Q &= F_s \cdot \lim_{n\to\infty} \frac{1}{n} H(Q_K(\boldsymbol{X}_q + \boldsymbol{Z})|\boldsymbol{Z}) \\ &\triangleq F_s \cdot \overline{H}(Q_K(X_q + Z)|Z) \end{aligned} \tag{7}$$

bits per second, where $\overline{H}(\cdot)$ denotes entropy rate per sample. It is simple to verify that since the K-blocks of $\boldsymbol{Z}$ are independent and since X is stationary, the conditional entropy per sample decreases monotonically with $n = K, 2K, 3K, \cdots$, and hence the limit above always exists. Note that by using a universal coding method that achieves the entropy for losslessly coding the quantizer output, the entire scheme becomes universal in the class of sources with prescribed band.

The error signal of the proposed coding scheme is $\hat{x}(t) - x(t)$, and hence the distortion, under MSE criterion, is

$$D = E\{(\hat{X}(t) - X(t))^2\}. \tag{8}$$

Note that the error signal is block-stationary (or cyclostationary) and thus in general, its expected distortion may vary periodically with time, with a cycle of KT_s. However, if desired, it can be forced to be stationary by choosing randomly, with uniform distribution, the initialization of the quantization block. Moreover, it is shown below that even without randomization the MSE is time-invariant and independent of the source.

To end this section and before we get to the detailed derivation, we summarize the main results of the paper. We shall be interested in comparing the performance of our scheme to the optimal performance, given by $R(D)$, the rate-distribution function of the source. Let $\rho = \rho(X, D, F_s, Q_K) = R_Q - R(D)$ be the redundancy, or the extra rate, which, in general, is a function of the source, the distribution level, the sampling rate, and the structure of the lattice quantizer. In this paper (Section IV) we show

$$\rho \to \rho_0(F_s, Q_K), \quad \text{as } D \to 0$$

for all "smooth" sources. Furthermore

$$\sup_{\{X,D\}} \rho \le B + \rho_0(F_s, Q_K)$$

where the supremum is over all sources (with bandwidth B and all distortion levels. Next, our main result (presented in Section V) is

$$\rho_0(F_s Q_K) \ge \rho_0(2B, Q_K) = B \log 2\pi G_K.$$

Roughly speaking, this inequality means that there is less, in coding rate, due to oversampling (i.e., sampling above the Nyquist rate). Nevertheless, as the lattice dimension of the quantizer increases, the loss due to oversampling decreases, as we show (in Section VI) that

$$\rho_0(F_s Q_K) \to), \quad \text{for all } F_s \ge 2B.$$

III. General Expressions for Rate-Distortion Performance

The rate of the sampling and quantization scheme is determined by the ECDQ block, and so it can be obtained from the results of [22], where the ECDQ performance in coding vector sources has been analyzed. Nevertheless, we provide, below, a simpler rederivation of the results of [22], and use it to get a general expression for the rate of the scheme presented in this paper.

Let the discrete dither signal, which is a concatenation of independent realization of $\boldsymbol{Z}_K$, be denoted $Z[n]$. Define $N_q[n] = \hat{X}_q[n] - X_q[n]$, i.e., N_q is the ECDQ error signal. We first recall the following theorem:

Theorem 1: N_q is independent of X_q and distributed as $-Z$ (i.e., when the lattice quantizer is symmetric, N_q is distributed as the dither.)

This theorem is well known at least for scalar dithered quantization (see [19] and also [10], p. 170]). For completeness it is proved in the Appendix, Subsection A, for the general lattice case. The theorem shows that as far as the input/output relations are considered, the ECDQ block is equivalent to the discrete additive noise channel, depicted in Fig. 1(b), whose input is $X_q[n]$ and its output is $\hat{X}_q[n] = X_q[n] + N_q[n]$. We show below that the scheme's distortion is determined from this input/output relation.

As for the rate, we now show in Theorem 2, that the entropy of the quantizer output, which defines the scheme's rate, is the mutual information between X_q and $\hat{X}_q$. Specifically, let $\boldsymbol{X}_1$,

$\hat{\boldsymbol{X}}_q$, $\boldsymbol{Z}$, and $\boldsymbol{N}_q$ denote blocks of length n as defined above. We claim the following theorem.

Theorem 2:

$$H(Q_K(\boldsymbol{X}_q+\boldsymbol{Z})|\boldsymbol{Z}) = I(\boldsymbol{X}_q;\, \hat{\boldsymbol{X}}_q) = I(\boldsymbol{X}_q;\, \boldsymbol{X}_q+\boldsymbol{N}_q) \quad (9)$$

where $I(\cdot;\cdot)$ denotes mutual information.

Proof: Observe that

$$\begin{aligned} H(Q_K(\boldsymbol{X}_q+\boldsymbol{Z})|\boldsymbol{Z}) &= H(Q_K(\boldsymbol{X}_q+\boldsymbol{Z})-\boldsymbol{Z}|\boldsymbol{Z}) \\ &= H(\hat{\boldsymbol{X}}_q|\boldsymbol{Z}). \end{aligned} \quad (10)$$

Since $\boldsymbol{X}_q$ and $\boldsymbol{Z}$ determine $\hat{\boldsymbol{X}}_q$, $H(\hat{\boldsymbol{X}}_q|\boldsymbol{X}_q, \boldsymbol{Z}) = 0$ and so

$$H(\hat{\boldsymbol{X}}_q|\boldsymbol{Z}) = H(\hat{\boldsymbol{X}}_q|\boldsymbol{Z}) - H(\hat{\boldsymbol{X}}_q|\boldsymbol{X}_q, \boldsymbol{Z}) = I(\boldsymbol{X}_q;\, \hat{\boldsymbol{X}}_q|\boldsymbol{Z}). \quad (11)$$

Now, by the chain rule for the mutual information

$$I(\boldsymbol{X}_q;\, \hat{\boldsymbol{X}}_q|\boldsymbol{Z}) = I(\boldsymbol{X}_q;\, \hat{\boldsymbol{X}}_q, \boldsymbol{Z}) - I(\boldsymbol{X}_q;\, \boldsymbol{Z}). \quad (12)$$

However, since we can also write $\boldsymbol{Z} = Q_K(\hat{\boldsymbol{X}}_q) - \hat{\boldsymbol{X}}_q$, $\boldsymbol{Z}$ can be expressed deterministically in terms of $\hat{\boldsymbol{X}}_q$, and so

$$I(\boldsymbol{X}_q;\, \hat{\boldsymbol{X}}_q, \boldsymbol{Z}) = I(\boldsymbol{X}_q;\, \hat{\boldsymbol{X}}_q).$$

Also, since $\boldsymbol{Z}$ is independent of $\boldsymbol{X}_q$, $I(\boldsymbol{X}_q;\, \boldsymbol{Z}) = 0$. Thus

$$I(\boldsymbol{X}_q;\, \hat{\boldsymbol{X}}_q|\boldsymbol{Z}) = I(\boldsymbol{X}_q;\, \hat{\boldsymbol{X}}_q).$$

Combining this with (10) and (11), leads to the desired relation (9). ∎

This theorem was originally proved in a different, more complicated, way in [22]. Note that the proof here holds for any distribution of $\boldsymbol{X}_q$, discrete or continuous. In light of this theorem, the asymptotic coding rate of the scheme, (7), in encoding the continuous time source $X(t)$, is given by the mutual information rate per second between the input and output of the discrete equivalent channel, i.e.

$$R_Q = F_s \cdot \lim_{n\to\infty} \frac{1}{n} I(\boldsymbol{X}_q;\, \hat{\boldsymbol{X}}_q) \triangleq F_s \cdot \overline{I}(X_q;\, \hat{X}_q). \quad (13)$$

Using Theorem 1, we may further write

$$\begin{aligned} R_Q &= F_s \cdot \overline{I}(X_q;\, X_q+N_q) \\ &= F_s \cdot \left(\overline{h}(X_q+N_q) - \frac{1}{2}\log(\epsilon/G_K)\right) \end{aligned} \quad (14)$$

where $\overline{h}$ denotes differential entropy rate per sample. The right-hand side of (14) follows by decomposing the mutual information rate into a different of entropy rates, and substituting

$$\overline{h}(N_q) = \frac{1}{K}\log V = \frac{1}{2}\log(\epsilon/G_K)$$

(see [22]). Since

$$\overline{h}(N_q) = \tfrac{1}{2}\log(\epsilon/G_K)$$

is finite, and

$$\overline{h}(X_q+N_q) \geq \overline{h}(N_q)$$

the existence of the rate limits in (7) and (13) is confirmed by (14).

We now consider the distortion in coding $X(t)$. Let

$$N(t) = \sum_n N_q[n] \frac{\sin \pi(t-nT_s)/T_s}{\pi(t-nT_s)/T_s} \quad (15)$$

be the output of a Discrete-to-Continuous (D/C) converter whose input in $N_q[n]$. We use a white lattice quantizer (see (5) above) and so the samples of N_q are uncorrelated and have an equal power $\epsilon = G_K V^{2/K}$. Thus the noise $N(t)$ is a *wide-sense-stationary process*, with a flat power spectral density

$$S_N(f) = \begin{cases} 2\epsilon T_s, & \text{for } f \leq F_s/2 \\ 0, & \text{otherwise} \end{cases} \quad (16)$$

over the frequency range $(0, F_s/2)$. Let $N_B(t)$ be the signal achieved by further low-passing the continuous-time noise $N(t)$ to the frequencies $f \leq B$. With these definitions, it follows from Theorem 1 that the error process $\hat{X}(t) - X(t)$ equals in distribution to $N_B(t) + X_B(t) - X(t)$, where $X_B(t)$ is the output of the anti-aliasing filter, as depicted in Fig. 1. Furthermore, since, by (1), the process $X_B(t)$ equals $X(t)$ in the mean-square sense, i.e.

$$E\{(X_B(t) - X(t))^2\} = 0 \quad (17)$$

we have

$$\begin{aligned} E\{(\hat{X}(t) - X(t))^2\} &= E\{N_B(t)^2\} + E\{(X_B(t) - X(t))^2\} \\ &= E\{N_B(t)^2\}. \end{aligned}$$

Summarizing all the above, the overall MSE of the scheme (8) is given by

$$D = E\{N_B(t)^2\} = \int_0^B S_N(f)\, df = \epsilon \cdot \frac{2B}{F_s} = 2\epsilon T_s B. \quad (18)$$

Observe from (18) that as long as ϵT_s is kept constant, i.e., a simple tradeoff is kept between the sampling rate and the quantization resolution, the MSE distortion is the same. Note, however, that this simple tradeoff is valid only if the lattice quantizer is white, although the additive noise channel model of the scheme and the coding rate formula (13) still hold in general.

IV. Analysis of the Coding Rate Formula

At any sampling period T_s and any resolution ϵ, the general expressions derived in the previous section provide the rate-distortion curve $R_Q(D)$ of the proposed coding scheme, for any given source. However, these expressions may be too complicated to calculate and are too generic to provide insight regarding, e.g., the tradeoff between the sampling rate and the quantizer resolution. In this section we identify the major factors which dominate the behavior of the coding rate as a function of the quantizer resolution and the sampling rate.

We analyze the coding scheme behavior at various sampling rates. Thus for a unified framework, our results are presented in terms of the *continuous-time* additive noise channel depicted in Fig. 2. In this channel, the noise $n(t)$ defined in (15) is added to the signal $x_B(t)$ obtained by pre-filtering the source, and the result, $\tilde{x}(t) = x_B(t) + n(t)$, is passed through a

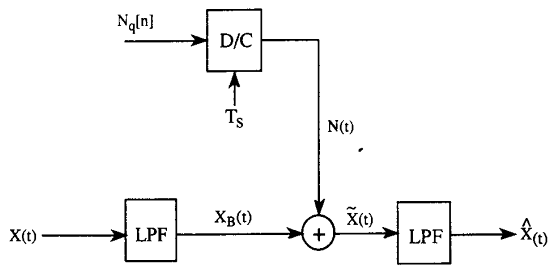


Fig. 2. Equivalent continuous-time channel for rate and distortion.

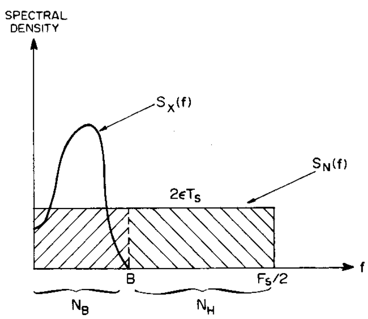


Fig. 3. X, N_B, and N_H and their spectra.

low-pass filter to yield the output, $\hat{x}(t) = x_B(t) + n_B(t)$. A sample function of the high-passed part of the noise which is filtered away is denoted $n_H(t) = n(t) - n_B(t)$. In Fig. 3 we show typical spectra of the source (X), the in-band noise (N_B), and the high-passed noise (N_H). Clearly, the equivalent continuous-time channel of Fig. 2 preserves the statistical relations between X, X_B, $\tilde{X}$, and $\hat{X}$, which are the continuous-time inputs and outputs of the coding scheme. In Theorem 3 below we further show that the coding rate may also be written in terms of mutual information rates between these continuous-time processes.

Before stating that theorem, we need to introduce some definitions and notations. First, we recall that there are several possible definitions for the mutual information rate between continuous valued processes (see, e.g., [16, p. 76] and [9, pp. 135–141]). In this paper we mostly use the so-called *Pinsker rate.* Let $X = \{X(t), -\infty < t < \infty\}$ and $Y = \{Y(t), -\infty < t < \infty\}$, be continuous-time processes with continuous values. Pinsker's rate is defined as

$$\overline{I}^{(g)}(X; Y) = \sup_{h,\, q_x,\, q_y} \frac{1}{h}\overline{I}(q_x(X^{(h)}); q_y(Y^{(h)})) \tag{19}$$

bits per second, where

$$X^{(h)} = \{X(nh), n = 0, \pm 1, \pm 2 \cdots\}$$

$$Y^{(h)} = \{Y(nh), n = 0, \pm 1, \pm 2 \cdots\}$$

$q_x(\cdot)$, $q_y(\cdot)$ denotes a time-invariant scalar quantizer with a *finite* number of levels, and $\overline{I}$ is the (regular) mutual information rate *per sample*, defined in (13), between the discrete-time processes $q_x(X^{(h)})$ and $q_y(Y^{(h)})$ which have discrete values. The supremum in (19) is taken over all possible sampling periods h and finite quantizers q_x and q_y. A similar definition applies for the Pinsker rate between *discrete-time* processes, where then we fix n to be the sampling period. For jointly stationary processes, Pinsker's rate always exists. Note that Pinsker's rate between processes in which each sample function is bandlimited (like $X_B(t)$ or $N(t)$), is equal to Pinsker's rate between the *sampled* processes after the appropriate normalization to bits per second. This property is one of the important features of the definition (19), and it follows directly from the fact that $\overline{I}^{(g)}$ satisfies the data processing theorem (see [16, p. 95, properties (6) and (7)]). This property enables us to associate the information rates in the discrete part of the coding system with the rates in its continuous part. It should be pointed out that the other definitions of the mutual information rate, also made in [16] under the names $\overline{I}$, $\tilde{I}$, and $\vec{I}$, lead to meaningless values (0 or ∞) for continuous-time bandlimited processes. Nevertheless, these other definitions are useful in the discrete-time case, and are utilized in certain cases below.

Second, we make the following definitions of a *nondegenerate source,* and a *smooth source:*

Definition 1: A source $X(t)$ is *nondegenerate* if the Nyquist sampled process of $X_B(t)$ has the "finite-gap information property" (see [9, sect. 6.4]), i.e.

$$I(X_B(0); X_B(-1/2B), X_B(-2/2B), \cdots) < \infty. \tag{20}$$

Furthermore, the source $X(t)$ is *smooth* if the Nyquist sampled process of $X_B(t)$ is smooth, i.e., its differential entropy rate exists and is finite

$$\overline{h}_x \triangleq \overline{h}(X_B(1/2B), X_B(2/2B), \cdots) > -\infty. \tag{21}$$

The first property above provides a key tool in our analysis, since the various definitions of the mutual information rate coincide for the Nyquist sampled process of a nondegenerate source (see [16, Theorem 7.4.2] and [9, Theorem 6.4.2]). The property of smoothness is important since it implies nondegeneration, and, as will be shown in the next subsection, it allows simple analysis in the low distortion limit.

For a Gaussian source

$$\overline{h}_x = \frac{1}{2B}\int_0^B \log(2\pi e B S_x(f))\, df$$

and the mutual information in (20) is

$$\tfrac{1}{2}\log 2\pi e \sigma_x^2 - \overline{h}_x.$$

Thus both conditions (20) and (21) become

$$\int_0^B \log S_x(f)\, df > -\infty. \tag{22}$$

In the general case, (22) is a necessary condition for smoothness since the Gaussian entropy upperbounds the source entropy.

We now return to the continuous-time channel of Fig. 2. In the following theorem we provide expressions for the scheme's coding rate in terms of the mutual information rates between signals in the equivalent channel:

Theorem 3:

$$R_Q = \overline{I}^{(g)}(X;\, \tilde{X}) \geq \overline{I}^{(g)}(X;\, \hat{X}) \tag{23}$$

with equality at the Nyquist sampling rate $F_x = 2B$. Furthermore, for a nondegenerate source at any sampling rate,

$$R_Q = \overline{I}^{(g)}(X;\, \hat{X}) + \overline{I}^{(g)}(N_B;\, N_H) - \overline{I}^{(g)}(\hat{X};\, N_H). \tag{24}$$

The proof is given in the Appendix, Subsection C.

Observe that from this theorem the scheme's coding rate can be written as the mutual information rate between X and $\tilde{X}$, but not between X and $\hat{X}$. Moreover, the lower bound in (23) is strict in some cases and so this theorem actually implies that the rate of the coding scheme is higher than the mutual information rate between the input signal and the reconstructed signal. Intuitively, the reason is that some extra bits are transferred since the scheme does not fully utilize the fact that at reconstruction the signal is filtered.

An alternative reason is that the information on the source that exists in the outband noise is ignored, due to the filter at reconstruction.

As for the more technical aspects of the theorem, we note that (24) is well defined: Pinsker's rate $\overline{I}^{(g)}$ always exists the rate

$$R_Q = \overline{I}^{(g)}(X;\, \tilde{X}) \geq \overline{I}^{(g)}(X;\, \hat{X})$$

is finite from (7) or (14), and as shown in the Appendix, Subsection B,

$$\overline{I}^{(g)}(N_B;\, N_H) \leq F_s/2 \log 2\pi e G_K$$

is finite as well. The nondegeneration condition required for (24) was needed technically for the proof, but we are not sure whether it is really a necessary condition.

We finally note that the proof of the theorem would have been very simple if we could apply naively the chain rule and other properties of the regular mutual information in (23) and (24). However, since these expressions are given in terms of mutual information rates of processes, a more complicated and careful derivation should be performed. Thus the detailed, and somewhat tedious proof, which utilizes all the definitions made above, is given in the Appendix, Subsection C.

In this section we shall also be interested in comparing the performance of our scheme to the optimal performance, given by the rate-distortion function of the source, defined as

$$R(D) = F_s \cdot \lim_{n\to\infty} R_n(D) \tag{25}$$

where $R_n(D)$ is the rate-distortion function per sample of the vector of n samples $X_q[1] \cdots X_q[n]$ of the sampled process X_q under square-error distortion measure. The definition (25) is actually the standard definition of the rate-distortion function of the sampled process X_q (see [1]). However, using the equivalent *process* definition of the rate-distortion function (see [9, sec. 10.6]), and a simple application of the data processing theorem for the Pinsker rate, it can be shown that $R(D)$ of (25) is equal to

$$R(D) = \inf_{\{U:\, E\{(X(t)-U(t))^2\} \leq D\}} \overline{I}^{(g)}(X;\, U) \tag{26}$$

bits per second, where U is jointly stationary with X. This last definition (26) is close to the "ϵ-entropy rate" of X defined in [12]. From (26) it is clear that $R(D)$ is indeed independent of the actual sampling rate of X, as we have mentioned in the Introduction.

In the rest of this section we further analyze the scheme's rate, and its excess rate over the rate-distortion function of the source in two cases. First, we consider the behavior at low distortion, and so this analysis is in the realm of high-resolution quantization theory. The second derivation provides a constant upper bound on the excess rate of the scheme over the source's rate-distortion function which holds for all distortion levels, and so it provides a worst case figure for the scheme's performance. Analyzing the effects of the sampling rate on the rate expressions is deferred to Section V.

A. Low-Distortion Behavior of the Coding Rate

The analysis is performed by identifying the main components in the coding rate formula (24), and observing that for smooth sources, at low distortion, there are essentially two terms. One term is the equivalent of Shannon's lower bound on the rate-distortion function, which is independent of the sampling rate but depends on the source and the distortion level. The second term, which is further analyzed at Section V, is independent of the source and the distortion level, but depends on the sampling rate.

We begin our analysis of (24) with the term $\overline{I}^{(g)}(X;\, \hat{X})$, whose behavior parallels that of the discrete-time ECDQ of [22]. Define

$$P_x = \frac{1}{2\pi e} 2^{2\overline{h}_x}$$

to be the entropy power (rate) of the Nyquist samples of $X_B(t)$ (see [1]), where $\overline{h}_x$ was defined in (21). Following [22], we define the "resolution measure" of the coding procedure with respect to the source as

$$r(D) \triangleq \overline{I}^{(g)}(N_B;\, N_B + X_B). \tag{27}$$

By definition, $r(D) \geq 0$. It is shown in the Appendix, Subsection D that for smooth sources $r(D) < \infty$ and $r(D) \to 0$ as $D \to 0$. Now for smooth sources we claim the following lemma.

Lemma 1:

$$\overline{I}^{(g)}(X;\, \hat{X}) = B \cdot \log\left(\frac{P_x}{D}\right) + \overline{\mathcal{D}}^{(g)}(N_B;\, N_B^*) + r(D). \tag{28}$$

The proof of Lemma 1 is given in the Appendix, Subsection E. The notation $\overline{\mathcal{D}}^{(g)}(N_B;\, N_B^*)$ is the "divergence from Gaussianity" of N_B in bits per unit time (see [23] and [9, cor. 7.4.3]), i.e., it is the divergence rate between the Nyquist sampled process of N_B and the Nyquist sampled process of the Gaussian process N_B^* having the same mean and spectrum as N_B. By the *divergence data processing theorem*

$$\overline{\mathcal{D}}^{(g)}(N_B;\, N_B^*) \leq \overline{\mathcal{D}}^{(g)}(N;\, N^*) = \frac{F_s}{2} \log 2\pi e G_K \tag{29}$$

with equality at the Nyquist sampling rate $F_s = 2B$ (see the Appendix, Subsection B and [24]). As a matter of fact, a tighter bound

$$\overline{\mathcal{D}}^{(g)}(N_B;\, N_B^*) \le B \log 2\pi e G_K$$

can be deduced from a generalization of the Entropy Power Inequality (EPI) shown in [23]. Now by [24], the optimal lattice quantizers satisfy $G_K^{\text{opt}} \to 1/2\pi e$ as $K \to \infty$, and so if we use optimal lattices, we get that as the dimension grows $\overline{\mathcal{D}}^{(g)}(N_B;\, N_B^*) \to 0$. We return to this point in Section VI below.

The term $B \cdot \log(P_x/D)$ in (28) is a lower bound, actually it is a generalization of the Shannon lower bound for the rate-distortion function of the source ([1, Theorem 4.6.5]), i.e.

$$\begin{aligned} R(D) \ge R_L(D) &= 2B \cdot \left[\overline{h}_x - \frac{1}{2}\log(2\pi e D)\right] \\ &= B \log\left(\frac{P_x}{D}\right). \end{aligned} \tag{30}$$

We now return to the rest of the terms in the coding rate formula (24). The term $\overline{I}^{(g)}(N_B;\, N_H)$ is independent of both the source and the distortion but may depend on the sampling rate, and at this point it will not be analyzed further. As for the term $\overline{I}^{(g)}(\hat{X};\, N_H)$, by the data-processing theorem, it is upper-bounded by the resolution measure, i.e.

$$\overline{I}^{(g)}(\hat{X};\, N_H) \le \overline{I}^{(g)}(\hat{X};\, N_B) = r(D)$$

and so it is negligible for small D.

In summary, we can combine (24), (28), and (30), and express the coding rate of the scheme for smooth sources as

$$R_Q(D) = \underbrace{R_L(D)}_{\text{indpt. of } F_s} + \underbrace{\overline{\mathcal{D}}^{(g)}(N_B;\, N_B^*) + \overline{I}^{(g)}(N_B;\, N_H)}_{\text{depend on } F_s,\ \text{indpt. of } X,\, D} + \underbrace{O(r(D))}_{\text{vanishes with } D}. \tag{31}$$

Furthermore, for smooth sources $R(D) - R_L(D) \to 0$, as $D \to 0$ (see [13]). Thus the low-distortion analysis of the coding scheme can be summarized by the following theorem, which is given in terms of the *redundancy* of the scheme $\rho(D) \triangleq R_Q(D) - R(D)$.

Theorem 4: For any nondegenerate source

$$\rho(D) \le \overline{\mathcal{D}}^{(g)}(N_B;\, N_B^*) + \overline{I}^{(g)}(N_B;\, N_H) + r(D). \tag{32}$$

Furthermore, for smooth sources

$$\rho(D) \to \overline{\mathcal{D}}^{(g)}(N_B;\, N_B^*) + \overline{I}^{(g)}(N_B;\, N_H), \qquad \text{as } D \to 0. \tag{33}$$

The expressions in (31) and in Theorem 4 above identify the quantity $\overline{\mathcal{D}}^{(g)}(N_B;\, N_B^*) + \overline{I}^{(g)}(N_B;\, N_H)$ as the component of the coding rate which depends on the sampling rate (but not on the source nor on the distortion). It turns out that as we increase the sampling rate, N_B approaches normality and so $\overline{\mathcal{D}}^{(g)}(N_B;\, N_B^*) \to 0$. On the other hand, $\overline{I}^{(g)}(N_B;\, N_H)$ increases. The detailed effects of increasing the sampling rate are discussed in Section V below.

We note that at Nyquist's sampling rate the expression (33) for the low-distortion redundancy becomes $\frac{1}{2}\log 2\pi e G_K$ bits per Nyquist sample, agreeing with well-known results from entropy-constrained high-resolution quantization theory (see [6] and [7]). Thus (33) suggests an extension of the high-resolution quantization theory to the case of oversampled sources.

So far our analysis focused on the low-distortion case for smooth sources, where the resolution measure vanishes. Next, we derive the continuous-time version of the universal "capacity bound" of [22], which holds for *all* distortion levels and *all* sources. Interestingly, the quantity $\overline{\mathcal{D}}^{(g)}(N_B;\, N_B^*) + \overline{I}^{(g)}(N_B;\, N_H)$ also dominates the behavior of this universal bound.

B. The Constant (Capacity) Bound

Consider again the equivalent channel, depicted in Fig. 2. Define by C the following capacity:

$$\begin{aligned} C &= F_s \cdot \sup_{\{X:\, E\{X^2(t)\} \le D\}} \overline{I}(X_q;\, X_q + N_q) \\ &= \sup_{\{X:\, E\{X^2(t)\} \le D\}} \overline{I}^{(g)}(X;\, \tilde{X}). \end{aligned} \tag{34}$$

This is the power-constrained capacity of the channel whose output is the signal before the low-pass filter in the output of the equivalent channel of Fig. 2. Since the noise power in the band $f \le B$ is D, and since the mutual information is invariant to scaling, the power constraint $E\{X^2(t)\} \le D$ in (34) actually means that the SNR in the effective band is restricted to be at most 1, or at most $2B/F_s$ in the entire band, and so C is independent of D. As claimed in the following theorem, the capacity (34) is an upper bound for the scheme's redundancy.

Theorem 5: For any source

$$\rho(D) = R_Q(D) - R(D) \le C. \tag{35}$$

The proof of this Theorem, which is similar to the proof of [22, Theorem 2], is given in the Appendix, Subsection F.

The capacity bound of Theorem 5 is tight for the high-distortion case, as it can be attained at high distortion by the source that achieves the capacity. To see this, observe that the power of this source (which in general may be block-stationary) is D, as the allowed distortion. Thus the rate distortion function of this source at distortion D is *zero* and so the redundancy is the quantizer rate which is the mutual information or, for this source, the capacity (34) of the channel. From this example we conclude that the capacity bound cannot be improved by another constant bound.

This theorem extends [22, Theorem 2], which can be applied only when operating at Nyquist's rate where the noise band is identical to the signal band. To assess the effect of higher sampling rate on the capacity we suppose that the capacity, given by (34), can be decomposed as in (24), and so we obtain

$$C_B \le C \le C_B + \overline{I}^{(g)}(N_B;\, N_H) \tag{36}$$

where

$$\begin{aligned} C_B &= \sup_{\{X:\, E\{X^2(t)\} \le D\}} \overline{I}^{(g)}(X;\, \hat{X}) \\ &= \sup_{\{X:\, E\{X^2(t)\} \le D\}} \overline{I}^{(g)}(X_B;\, X_B + N_B) \end{aligned} \tag{37}$$

is the constrained capacity (in bits per unit time) of a channel with a band-limited additive noise.

Note that for Nyquist's rate we have $C_B \approx 1/2 \log 4\pi e G_K$ bits per sample, which is the capacity bound of [22]. Following [22, Appendix C], C_B can be further bounded as

$$B \log\left(1 + 2^{\frac{1}{B}\overline{\mathcal{D}}^{(g)}(N_B;\, N_B^*)}\right) \leq C_B \leq B + \overline{\mathcal{D}}^{(g)}(N_B;\, N_B^*). \tag{38}$$

When $\overline{\mathcal{D}}^{(g)}(N_B;\, N_B^*) \to 0$ which happens, as noted above, at high sampling rate and at large lattice dimension, we get $C_B = B$ bits per unit time.

Equations (36) and (38) can be combined and we get the following desired upper bound:

$$\rho(D) \leq B + \overline{\mathcal{D}}^{(g)}(N_B;\, N_B^*) + \overline{I}^{(g)}(N_B;\, N_H). \tag{39}$$

By comparing (33) to (39) we observe that for smooth sources the redundancy at low distortion is smaller by at most B bits per unit time (or half a bit per Nyquist sample) than the redundancy at high distortion. These results have the same flavor as our bounds for vector sources in [22].

V. The Effect of Increasing the Sampling Rate

We have already discussed the effect of the sampling rate on the distortion of the coding scheme. As noted in Section III, for square error distortion $D = 2\epsilon T_s$, and so there exists a simple tradeoff between the sampling period (T_s) and the quantizer resolution (given by ϵ, the second moment of its basic cell) in determining the distortion. We now examine the effect of increasing the sampling rate on the coding rate R_Q. To set a common ground for comparison, we assume in the analysis that while the sampling rate increases (i.e., T_s decrease), $2\epsilon T_s$ is kept constant, and its value is determined by the allowed distortion. This constant is the spectral level of the noise $N(t)$ (see (16)). The quantizer choice determines the noise distribution (for example, for scalar quantizer the samples of N_q are uniformly distributed), and is also assumed fixed. The sampling rate, whose variation is examined, determines the fraction of the entire band that is occupied by the in-band noise $N_B(t)$.

Since the distortion is fixed, the rate-distortion function of the source is fixed, and the effect of increasing the sampling rate, on the coding rate, is reflected in the scheme's redundancy $\rho(D) = R_Q(D) - R(D)$. In the high-distortion case, this redundancy is close to the capacity bound in (39), while in the medium- and low-distortion cases, the effective redundancy expressions are (32) and (33). All these expressions are valid when sampling at Nyquist's rate or faster. As pointed out above, in all these expressions for the redundancy there are terms which depend on D which vanish (for smooth sources) as $D \to 0$ and become B (in the bound) for high distortion, and there are the two common terms, $\overline{\mathcal{D}}^{(g)}(N_B;\, N_B^*)$ and $\overline{I}^{(g)}(N_B;\, N_H)$, which are independent of D and the source. The analysis of these two terms that are strongly affected by the sampling rate, sheds light on the effect of oversampling on the coding rate of the scheme. This analysis is given in the following theorem, which is rigorously proved for the scalar quantizer case ($K = 1$) and conjectured for the general case.

Theorem 6: For any sampling rate

$$\begin{aligned}\frac{F_s}{2} \cdot \log 2\pi e G_K &\geq \overline{\mathcal{D}}^{(g)}(N_B;\, N_B^*) + \overline{I}^{(g)}(N_B;\, N_H) \\ &\geq B \cdot \log 2\pi e G_K.\end{aligned} \tag{40}$$

Note that at the Nyquist rate where $F_s = 2B$, both upper and lower bounds coincide and so

$$\overline{\mathcal{D}}^{(g)}(N_B;\, N_B^*) + \overline{I}^{(g)}(N_B;\, N_H) = B \cdot \log 2\pi e G_K.$$

Thus the lower bound can be interpreted as the value of $\overline{\mathcal{D}}^{(g)}(N_B;\, N_B^*) + \overline{I}^{(g)}(N_B;\, N_H)$ at the Nyquist rate. The interesting implication of the theorem and this observation is that when the distortion is kept constant the coding rate of the scheme operating at a sampling rate higher than Nyquist's rate is *larger* than the rate of the scheme operating at Nyquist's rate.

Proof: In the Appendix, Subsection B we show that

$$\begin{aligned}\overline{I}^{(g)}(N_B;\, N_H) = \overline{\mathcal{D}}^{(g)}&(N;\, N^*) \\ &- \overline{\mathcal{D}}^{(g)}(N_B;\, N_B^*) \\ &- \overline{\mathcal{D}}^{(g)}(N_H;\, N_H^*)\end{aligned}$$

and thus

$$\begin{aligned}\overline{\mathcal{D}}^{(g)}(N_B;\, N_B^*) &+ \overline{I}^{(g)}(N_B;\, N_H) \\ &= \overline{\mathcal{D}}^{(g)}(N;\, N^*) - \overline{\mathcal{D}}^{(g)}(N_H;\, N_H^*).\end{aligned} \tag{41}$$

The upper bound in (40) is obtained by substituting

$$\overline{\mathcal{D}}^{(g)}(N;\, N^*) = F_s \cdot \tfrac{1}{2} \log 2\pi e G_K$$

into (41), and utilizing the nonnegativity of the divergence.

To obtain the lower bound in (40), we use the generalization of the Entropy Power Inequality (EPI), developed in [23], which provides the following upper bound (see [23, eq. (21)] for the divergence from Gaussianity of an i.i.d. vector $\boldsymbol{N} = N_1, \cdots, N_n$ multiplied by a noninvertible matrix A:

$$\frac{1}{m}\mathcal{D}(A\boldsymbol{N};\, A\boldsymbol{N}^*) \leq \frac{1}{n}\mathcal{D}(\boldsymbol{N};\, \boldsymbol{N}^*) \tag{42}$$

where m is the rank of the matrix A, $\mathcal{D}(\cdot;\,\cdot)$ is the divergence, and $\boldsymbol{N}^*$ is a Gaussian vector having the same second moments as $\boldsymbol{N}$. A similar relation can be stated for a process N interpolated from an i.i.d. sequence using the interpolation formula of (15)

$$\frac{1}{m}\overline{\mathcal{D}}^{(g)}(\mathcal{A}\{N\};\, \mathcal{A}\{N^*\}) \leq \frac{1}{n}\overline{\mathcal{D}}^{(g)}(N;\, N^*) \tag{43}$$

where $\mathcal{A}$ is a noninvertible linear time-invariant transformation, n is the number of degrees of freedom per second of N (given by $F_s = 1/T_s$ of the interpolation formula), and m is the number of degrees of freedom per second of the filtered process $\mathcal{A}\{N\}$. Now, N_H is the output of a noninvertible high-pass filter whose input is the process N interpolated from an i.i.d. sequence. The number of degrees of freedom per second of N_H is $F_s - 2B$. Thus

$$\overline{\mathcal{D}}^{(g)}(N_H;\, N_H^*) \le \frac{F_s - 2B}{F_s} \cdot \overline{\mathcal{D}}^{(g)}(N;\, N^*) = (F_s - 2B) \cdot \frac{1}{2} \log 2\pi e G_K \text{ (bits per second)}. \tag{44}$$

Combining (44) with the expressions above proves the lower bound in (40).

Note that since the generalization of the EPI was proved for a process with i.i.d. samples, it can be used only for $K = 1$, and the lower bound in (40) is proved only for the scalar ECDQ case. However, we conjecture that this generalization can be extended to i.i.d. K-blocks, so that the lower bound in (40) holds for general lattice ECDQ's. ■

To examine this theorem, we have explicitly calculated the terms

$$\overline{\mathcal{D}}^{(g)}(N_B,\, N_B^*) \triangleq \overline{\mathcal{D}}(F_s)$$

and

$$\overline{I}^{(g)}(N_B;\, N_H) \triangleq \overline{I}^{(g)}(F_s)$$

for the uniform scalar quantizer case, at a few sampling rate examples. For the Nyquist sampling rate $\overline{\mathcal{D}}(2B) \approx 0.254 \cdot 2B$ while $\overline{I}^{(g)}(2B) = 0$. As the sampling rate is increased by a factor of 2, $\overline{\mathcal{D}}(4B) \approx 0.033 \cdot 2B$ and it becomes (approximately) $0.009 \cdot 2B$ as the sampling rate is increased by a factor of 3. The term $\overline{I}(F_s)$ is approximately $0.44 \cdot 2B$ and $0.64 \cdot 2B$ as the sampling rate increases by factors of 2 and 3, respectively. We see that, indeed, in these examples, the rate of the coding scheme increases, as the sampling rate increases, even when the quantizer resolution is reduced to keep the same distortion. These calculations may also point out that the lower bound in (40) is loose, i.e., at high sampling rate the coding efficiency of the scheme is even worse than what is implied by the lower bound in (40).

VI. Large Lattice Dimension: Equivalent AWGN Channel

The expressions for the rate distortion of the proposed scheme, its redundancy, and the tradeoff relation between the sampling rate and the quantization accuracy would become simple, if we could have assumed that the additive noise in the equivalent channel of Fig. 2 is Gaussian, with a flat spectrum of level $2\epsilon T_s$, and independent of the source. In accordance with the notation above, this additive Gaussian noise is denoted by $N^*(t)$, while the additive noise in the output, after bandpass filtering, which is Gaussian and bandlimited to $0 \le f \le B$, is denoted $N_B^*(t)$. The MSE distortion is $2\epsilon T_s \cdot B$, the variance of $N_B^*(t)$. Since in the Gaussian case the components of $N^*(t)$ in the passband and the stopband are independent, the rate

$$\overline{I}^{(g)}(X_B;\, X_B + N^*) \triangleq R_Q^*(D) \tag{45}$$

is equal to $\overline{I}^{(g)}(X_B;\, X_B + N_B^*)$, and so as long as $\epsilon \cdot T_s$ is kept constant we get the same rate.

As discussed in [24], the proposed quantization scheme becomes equivalent to an Additive White Gaussian Noise (AWGN) channel in the limit as the lattice dimension becomes large, at any fixed sampling rate. To see this, we use a result, proved by Poltyrev [17], asserting that the normalized second moment of the optimal lattice quantizer satisfies,

$$\lim_{K\to\infty} G_K^{\text{opt}} = \frac{1}{2\pi e} \approx 0.058823 \tag{46}$$

where G_K^{opt} is the minimal value of G_K over all lattices of dimension K. This result implies

$$\overline{\mathcal{D}}^{(g)}(N^{(K)};\, N^*) = F_s \cdot \frac{1}{2} \log 2\pi e G_K^{\text{opt}} \to 0, \qquad \text{as } K \to \infty \tag{47}$$

i.e., the quantization noise $N^{(K)}$ converges to Gaussianity in the *divergence sense*. Throughout this section we denote by superscript (K) the corresponding terms when an optimal K-dimensional lattice quantizer is used.

Combining (46) with (40), implies immediately that

$$\overline{\mathcal{D}}^{(g)}(N_B^{(K)};\, N_B^*) + \overline{I}^{(g)}(N_B^{(K)};\, N_H^{(K)}) \to 0$$

for any sampling rate, as $K \to \infty$. Thus the main results of Section III can be simplified in the following way: (31), which expresses the rate at low distortion for smooth sources, becomes in the limit

$$\lim_{K\to\infty} R_Q^{(K)}(D) = R_L(D) + O(r(D)). \tag{48}$$

Equations (36) and (38) expressing the "capacity bound" and valid for all sources and distortion levels, become in the limit

$$\lim_{K\to\infty} R_Q^{(K)}(D) - R(D) \le \lim_{K\to\infty} C^{(K)} = B. \tag{49}$$

The limits in both (48) and (49) are the expressions that would have resulted if the scheme was an AWGN channel.

Finally, it should be noted that a stronger claim

$$\lim_{K\to\infty} R_Q^{(K)}(D) \le R_Q^*(D) \tag{50}$$

which holds for all smooth bandlimited sources, with equality if the source is Gaussian, can be shown from the results in [24]. In other words, for Gaussian sources, at any distortion level, our scheme is asymptotically equivalent, from the coding rate point of view, to an AWGN channel (for millimeter Gaussian sources, it may even be better). This equivalence implies that if we change the sampling rate, but change with it the quantizer resolution so that ϵT_s is kept constant, i.e., we keep the simple tradeoff that maintains the distortion fixed, the equivalent AWGN channel remains the same, and so the coding rate is also kept constant.

In the asymptotic Gaussian noise case it is easy to calculate the scheme's performance for Gaussian sources. For example, suppose our source has a power spectrum $S_x(f)$. Then, the rate of the coding scheme is given by

$$R_Q^*(D) = \int_0^B \log\left(1 + \frac{S_x(f)}{2\epsilon T_s}\right) df = \int_0^B \log\left(1 + \frac{S_x(f)}{D/B}\right) df \tag{51}$$

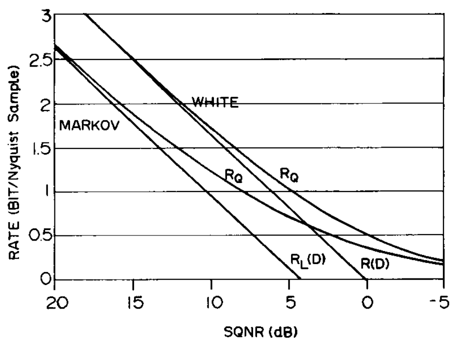


Fig. 4. Rate-distortion of the coding scheme for flat spectrum and Gauss–Markov sources compared with optimal performance.

where $D = \epsilon \cdot T_s \cdot 2B$ is the square error distortion and the rate is measured in bits per seconds. When the source has a flat spectrum with level σ_x^2/B, the rate of our scheme becomes $B \cdot \log(1 + (\sigma_x^2/D))$. The source's rate-distortion function is $B \cdot \log(\sigma_x^2/D)$. Thus the scheme's redundancy is

$$\rho(D) = B \cdot \log\left(1 + \frac{D}{\sigma_x^2}\right). \tag{52}$$

This redundancy expression holds for all distortion levels. At high distortion, $D/\sigma_x^2 \approx 1$ and we get $\rho = B$, i.e., 0.5 bits per Nyquist sample. This is also the capacity of an additive Gaussian noise channel when the input variance equals to the noise variance. At low distortion levels, where $D/\sigma_x^2 \to 0$, we find, as expected, that the redundancy approaches zero.

Fig. 4 summarizes the examples discussed above. It shows $R_Q(D)$, the performance of our scheme, for two sources: a Gaussian source with flat spectrum, and a first-order Gauss–Markov source for which the ratio of the -3 dB bandwidth (B_0) to the entire band (B) is 0.1. For comparison, we have plotted the rate-distortion function of the flat source, and the Shannon lower bound on the rate-distortion function of the Gauss–Markov source (since its rate-distortion function is hard to calculate). All plots are given as a function of $SQNR = \sigma_x^2/D$, (given in decibels) and normalized to bits per Nyquist's sample.

VII. Summary

Concluding the paper, we point out its main observation. When we tradeoff the sampling rate and the quantizer resolution, keeping the quantity ϵT_s (the quantizer second moment times the sampling rate) constant, we get the same distortion but the coding rate is increased. Thus if one prefers to use a simple, low-resolution, quantizer (say 1-bit quantizer) at the expense of higher sampling rate, which has some practical advantages, he should be aware that the overall bit rate is larger. This paper provides bounds on the resulting excess bit rate.

The results of this paper could have been presented for vector sources. The analogous of a band-limited process, sampled at a rate higher than Nyquist's rate, is a vector source whose components have linear deterministic relation, or in the mean-square sense, its correlation matrix does not have full rank. The analogous "oversampling ratio" is n/m, where n is the dimension of the vector source and m is the rank of its correlation matrix. Now, suppose this vector source is encoded by an ECDQ. Following the main observation of the paper, the coding rate would become smaller if, prior to coding, the source is projected over the (minimal) linear subspace where its energy is concentrated. This is, of course, analogous to coding the Nyquist sampled process in the bandlimited source case.

An oversampled but reduced resolution quantizer leads to another practical problem. Consider the tradeoff relation $\epsilon \cdot T_s$. We realize that since ϵ is proportional to the square of the quantizer step size, in order to save one bit in quantization, i.e., to use half the number of quantization levels, ϵ is increased by 4, and so the sampling rate must increase by 4 to get the same distortion. Thus without entropy coding the rate may increase by much more than what is implied by the results of this paper. For example, if we use 2 levels (1 bit) instead of 64 levels (6 bits) of A/D, we must increase the sampling rate by $4^5 = 1024$, getting 1024 bits for each original 6 bits at the A/D output. This increase in rate disappears later, after the lossless encoder. This observation emphasizes the important role played by the entropy encoder.

Note that in sigma–delta techniques [2] the rate increase at the A/D output is usually smaller than the increase mentioned above, since some of the lossless encoding is performed "on the fly," by filtering and prediction. Nevertheless, using the ECDQ does have in principle an advantage over sigma–delta methods. As expected, if sigma–delta coding with multiple bits is used, the distortion decreases exponentially with the number of bits, but the distortion decreases only polynomially (see [2]) as the sampling rate (and as a result the bit rate) increases. On the other hand, in ECDQ as the sampling rate increases, using the same quantizer, the distortion decreases exponentially with the bit rate since the performance of the scheme is at most a constant away from the source's rate-distortion function, and so it has the same behavior of R as a function of D when $D \to 0$. Again, this advantage comes from the fact that we use entropy coding and so an increase in the bit rate of the ECDQ might correspond to a much higher increase in the sampling rate.

Yet another word of caution should be mentioned. The ECDQ and our entire analysis assume a "subtractive dither," i.e., we assumed that the decoder has an access to the dither used by the encoder, and can subtract it. This might be cumbersome in some practical cases.

Finally, we note that the rate we calculated is the conditional entropy of the quantizer output. If we use a universal entropy coder that estimates the probabilities of the quantizer output, conditioned on the dither, we must use a discrete-valued dither realization. This will lead to an additional approximation of the theory derived in this paper.

Appendix

A. *Proof of Theorem 1*

Consider first a K-block of the error vector $\hat{\boldsymbol{X}}_K - \boldsymbol{X}_K$, and examine the conditional probability distribution function of $\hat{\boldsymbol{X}}_K - \boldsymbol{X}_K$ given $\boldsymbol{X}_K$, i.e.

$$\Pr\{\hat{\boldsymbol{X}}_K - \boldsymbol{X}_K \le \boldsymbol{\alpha} | \boldsymbol{X}_K\} = \Pr\{Q_K(\boldsymbol{X}_K + \boldsymbol{Z}_K) - (\boldsymbol{X}_K + \boldsymbol{Z}_K) \le \boldsymbol{\alpha} | \boldsymbol{X}_K\} \quad \text{(A1)}$$

where, in general $\Pr\{\boldsymbol{\beta} \le \boldsymbol{\alpha}\}$ means $\Pr\{\beta_1 \le \alpha_1, \cdots, \beta_K \le \alpha_K\}$ and $\{\alpha_i\}$, $\{\beta_i\}$ are the components of the K-dimensional vectors $\boldsymbol{\beta}$, $\boldsymbol{\alpha}$. We observe that the function $e(\boldsymbol{t}) \triangleq Q_K(\boldsymbol{t}) + \boldsymbol{t}$ has the lattice periodicity in the sense that $e(\boldsymbol{t}) = e(\boldsymbol{t} + \boldsymbol{l}_i)$ where $\boldsymbol{l}_i \in \mathcal{L}$ is a lattice point. Furthermore, the indicator function

$$I_{\boldsymbol{\alpha}}(\boldsymbol{t}) = \begin{cases} 1, & \text{if } e(\boldsymbol{t}) \le \boldsymbol{\alpha} \\ 0, & \text{otherwise} \end{cases}$$

has also the lattice periodicity. Now, we can write

$$\Pr\{\hat{\boldsymbol{X}}_K - \boldsymbol{X}_K \le \boldsymbol{\alpha} | \boldsymbol{X}_K\} = E_Z\{I_{\boldsymbol{\alpha}}(\boldsymbol{X}_K + \boldsymbol{Z}_K) | \boldsymbol{X}_K\} = \frac{1}{V}\int_{P_0} I_\alpha(\boldsymbol{X}_K + \boldsymbol{t})\, d\boldsymbol{t}. \quad \text{(A2)}$$

Since $I_{\boldsymbol{\alpha}}(\cdot)$ is lattice-periodic and the integral in (A2) is over the lattice cell, then, (A2) and so (A1), representing the probability of the error vector, do not depend on the value of $\boldsymbol{X}_K$, i.e., the error vector is statistically independent of $\boldsymbol{X}_K$.

As for the distribution of $\hat{\boldsymbol{X}}_K - \boldsymbol{X}_K$, since (A1) has the same value for each $\boldsymbol{X}_K$ we may choose $\boldsymbol{X}_K = \boldsymbol{0}$ and, since $Q_K(\boldsymbol{z}_K) = \boldsymbol{0}$, $\forall \boldsymbol{z}_K \in P_0$, we get

$$\Pr\{\hat{\boldsymbol{X}}_K - \boldsymbol{X}_K \le \boldsymbol{\alpha}\} = \Pr\{Q_K(\boldsymbol{Z}_K) - \boldsymbol{Z}_K \le \boldsymbol{\alpha}\} = \Pr\{-\boldsymbol{Z}_K \le \boldsymbol{\alpha}\} \quad \text{(A3)}$$

i.e., the error vector $\hat{\boldsymbol{X}}_K - \boldsymbol{X}_K$ is distributed as $-\boldsymbol{Z}_K$.

Since the dither is drawn independently for each K-block this result is easily extended to concatenation of K-blocks, and the theorem follows.

B. Existence and Decomposition of $\overline{I}^{(g)}(N_B;\, N_H)$

In this part of the Appendix we show that the Nyquist sampled processes of $N_B(t)$ and $N_H(t)$ are smooth (as defined in (21)), and that their mutual Pinsker rate $\overline{I}^{(g)}(N_B;\, N_H)$ is finite. For that, let

$$N_B^{(nq)}(t) = N_B\left(\frac{\lceil 2Bt \rceil}{2B}\right)$$

denote the "sample-and-hold" process, attained from $N_B(t)$ by sampling at Nyquist's rate and interpolating by a constant between the samples, and let $\boldsymbol{N}_B^{(T)}$ denotes the vector of Nyquist samples of $N_B(t)$ in the time interval $[0, T)$, i.e.

$$\boldsymbol{N}_B^{(T)} = N_B(1/2B) \cdots N_B(n/2B), \qquad \text{where } n = \lfloor 2BT \rfloor. \quad \text{(A4)}$$

Similarly, we define the processes $N_H^{(nq)}$ and $N^{(nq)}$, and the vectors $\boldsymbol{N}_H^{(T)}$ and $\boldsymbol{N}^{(T)}$ as the sampled processes and vectors of $N_H(t)$ and $N(t)$, sampled at the corresponding Nyquist rates $F_s - 2B$ and F_s, respectively, where for $N_H^{(nq)}$ and $\boldsymbol{N}_H^{(T)}$ we assume that $N_H(t)$ is down-converted to baseband prior to sampling. Since every realization of N_B and N_H is strictly bandlimited, they are completely determined by their Nyquist samples. Thus by the data processing theorem

$$\overline{I}^{(g)}(N_B;\, N_H) = \overline{I}^{(g)}(N_B^{(nq)};\, N_H^{(nq)}).$$

Now, assume for a moment that $N_B^{(nq)}$ has the finite-gap information property, i.e.,

$$I(N_B(0);\, N_B(-1/2B),\, N_B(-2/2B), \cdots) < \infty.$$

Under this assumption, we may replace the Pinsker rate $\overline{I}^{(g)}$ by

$$\overline{I}(N_B^{(nq)};\, N_H^{(nq)}) = \lim_{T\to\infty} \frac{1}{T} I(\boldsymbol{N}_B^{(T)};\, \boldsymbol{N}_H^{(T)}) \quad \text{(A5)}$$

see [9, Theorem 6.4.2.]. Now, the mutual information between the vectors $\boldsymbol{N}_B^{(T)}$ and $\boldsymbol{N}_H^{(T)}$ may be decomposed into a sum of entropies, according to the identity $I(A;\, B) = h(A) + h(B) - h(A,\, B)$. However, this decomposition is valid only if the differential entropies are *finite*. Thus we have to show that the limits in

$$\lim_{T\to\infty} \frac{1}{T} h(\boldsymbol{N}_B^{(T)}) + \lim_{T\to\infty} \frac{1}{T} h(\boldsymbol{N}_N^{(T)}) - \lim_{T\to\infty} \frac{1}{T} h(\boldsymbol{N}_B^{(T)},\, \boldsymbol{N}_H^{(T)}) \quad \text{(A6)}$$

exist and are *finite*. Furthermore, if

$$\lim_{T\to\infty} \frac{1}{T} h(\boldsymbol{N}_B^{(T)})$$

exists and is finite, it implies that the Nyquist sampled process of $N_B(t)$ is *smooth* and hence $N_B^{(nq)}$ indeed has the finite-gap information property and our assumption is true.

We first show that the third term in (A6) exists and is finite. Consider the divergence $\overline{\mathcal{D}}^{(g)}(N_B,\, N_H;\, N_B^*,\, N_H^*)$. Since it is invariant under invertible transformation of its arguments we have

$$\overline{\mathcal{D}}^{(g)}(N_B,\, N_H;\, N_B^*,\, N_H^*) = \overline{\mathcal{D}}^{(g)}(N;\, N^*) = F_s \cdot \tfrac{1}{2}\log(2\pi e G_K)$$

i.e., it is finite. Consider also

$$\lim_{T\to\infty} T^{-1} h(\boldsymbol{N}_B^{*(T)},\, \boldsymbol{N}_B^{*(T)}).$$

Since N_B and N_H are uncorrelated, the Gaussian vectors $\boldsymbol{N}_B^{*(T)}$ and $\boldsymbol{N}_H^{*(T)}$ are independent, and we can write

$$\begin{aligned} &\lim_{T\to\infty} \frac{1}{T} h(\boldsymbol{N}_B^{*(T)},\, \boldsymbol{N}_H^{*(T)}) \\ &= \lim_{T\to\infty} \frac{1}{T}[h(\boldsymbol{N}_B^{*(T)}) + h(\boldsymbol{N}_H^{*(T)})] \\ &= 2B \cdot \frac{1}{2}\log\left(2\pi e \frac{2B}{F_s}\epsilon\right) + (F_s - 2B) \\ &\quad \cdot \frac{1}{2}\log\left(2\pi e \frac{F_s - 2B}{F_s}\epsilon\right). \end{aligned} \quad \text{(A7)}$$

Now, since we can formally write

$$\overline{\mathcal{D}}^{(g)}(N_B,\, N_H;\, N_B^*,\, N_H^*) = \lim_{T\to\infty} \frac{1}{T}[h(\boldsymbol{N}_B^{*(T)},\, \boldsymbol{N}_H^{*(T)}) - h(\boldsymbol{N}_B^{(T)},\, \boldsymbol{N}_H^{(T)})] \quad \text{(A8)}$$

and since $\overline{\mathcal{D}}^{(g)}(N_B, N_H; N_B^*, N_H^*)$ is finite, then either both terms in the right-hand side of (A8) are undefined, or both terms are defined and finite. Since we just showed that

$$\lim_{T\to\infty} T^{-1}h(\boldsymbol{N}_B^{*(T)}, \boldsymbol{N}_H^{*(T)})$$

exists and finite, we conclude that

$$\lim_{T\to\infty} T^{-1}(\boldsymbol{N}_B^{(T)}, \boldsymbol{N}_H^{(T)})$$

is finite as well.

Now looking back at (A6), we note that it is a mutual information and so it is nonnegative. As just shown

$$\lim_{T\to\infty} T^{-1}h(\boldsymbol{N}_B^{(T)}, \boldsymbol{N}_H^{(T)})$$

exists; thus the other two terms of (A6) which are the entropies associated with its components also exist. In addition, these entropies have a finite upper bound—the entropies of the Gaussian processes N_B^* and N_H^*. But since

$$\lim_{T\to\infty} T^{-1}h(\boldsymbol{N}_B^{(T)}, \boldsymbol{N}_H^{(T)})$$

is finite

$$\lim_{T\to\infty} T^{-1}h(\boldsymbol{N}_B^{(T)})$$

and

$$\lim_{T\to\infty} T^{-1}h(\boldsymbol{N}_H^{(T)})$$

must be finite as well.

We finally point out that as in the relations above, we may express the divergence as a difference between entropies and can straightforwardly see that

$$\overline{I}^{(g)}(N_B; N_H) = \overline{\mathcal{D}}^{(g)}(N; N^*) - \overline{\mathcal{D}}^{(g)}(N_B; N_B^*) - \overline{\mathcal{D}}^{(g)}(N_H; N_H^*). \quad \text{(A9)}$$

C. Proof of Theorem 3

We begin with the first part, i.e., proving (23). We first show that the coding rate satisfies

$$R_Q \stackrel{(a)}{=} F_s \cdot \overline{I}(X_q; \hat{X}_q) \stackrel{(b)}{=} \overline{I}^{(g)}(X_q; \hat{X}_q) \stackrel{(c)}{=} \overline{I}^{(g)}(X; \tilde{X}). \quad \text{(A10)}$$

Our convention is that $\overline{I}^{(g)}$ is always measured in bits per second; hence, whenever $\overline{I}^{(g)}$ has a discrete-time argument (e.g., X_q or $\hat{X}_q$ in the equation above), it is considered as a continuous-time "sample-and-hold" process, having a constant value between two consecutive sample time points. It is important to make this convention, since, in some cases, the discrete-time processes we consider are obtained by different sampling rates.

Now, the equality (a) in (A10) follows from Theorem 2. To obtain equality (b), we show next that $\hat{X}_q$ is smooth, and thus it has the "finite-gap information property" (20) under which Pinsker's rate $\overline{I}^{(g)}$ and the regular mutual information rate $\overline{I}$ coincide. For that, we use well-known properties of the differential entropy to write

$$\begin{aligned} \infty > \tfrac{1}{2}\log 2\pi e(\sigma_x^2 + \epsilon) &\geq h(\hat{X}_{q^0}) \\ &\geq h(\hat{X}_{q^0} | \hat{X}_{q-1}, \hat{X}_{1-2}, \cdots) \\ &= \overline{h}(\hat{X}_q) = \overline{h}(X_q + N_q) \geq \overline{h}(N_q) \\ &= \tfrac{1}{2}\log(\epsilon/G_K) > -\infty. \end{aligned} \quad \text{(A11)}$$

This, together with the fact that $I(\hat{X}_{q^0}; \hat{X}_{q-1}, \hat{X}_{q-2}, \cdots) = h(\hat{X}_{q^0}) - h(\hat{X}_{q^0}|\hat{X}_{q-1}, \hat{X}_{1-2}, \cdots)$, leads to the desired condition (20). Finally, equality (c) is implied by the following sandwich argument: On one hand, observe that by construction $X \to X_q \to \hat{X}_q \to \tilde{X}$ form a Markov chain, and so, by the data-processing theorem, which as discussed above is satisfied by the Pinsker rate $\overline{I}^{(g)}(X_q; \hat{X}_q) \geq \overline{I}^{(g)}(X; \tilde{X})$. On the other hand, X_q is fully determined by X, and $\hat{X}_q$ is fully determined by $\tilde{X}$, implying, again by the data-processing theorem, $\overline{I}^{(g)}(X_q; \hat{X}_q) \leq \overline{I}^{(g)}(X; \tilde{X})$. Notice that in the derivation above we did not require the source to be bandlimited.

The lower bound in (23) follows from the data-processing theorem and the fact that $X \to \tilde{X} \to \hat{X}$ form a Markov chain. In the special case $F_x = 2B$ we have $\tilde{X} = \hat{X}$, and the inequality becomes equality.

We now turn to prove (24). We write the following chain of equalities:

$$\begin{aligned} R_Q &\stackrel{(a)}{=} \overline{I}^{(g)}(X_q; \hat{X}_q) \\ &\stackrel{(b)}{=} \overline{I}^{(g)}(X_B^{(nq)}; \hat{X}_q) \\ &\stackrel{(c)}{=} \tilde{I}(X_B^{(nq)}; \hat{X}) \\ &\stackrel{(d)}{=} \tilde{I}(X_B^{(nq)}; \hat{X}, N_H) \\ &\stackrel{(e)}{=} \tilde{I}(X_B^{(nq)}; \hat{X}) + \tilde{I}(X_B^{(nq)}; N_H | \hat{X}) \\ &\stackrel{(f)}{=} \overline{I}^{g}(X_B^{(nq)}; \hat{X}) + \overline{I}^{(g)}(X_B^{(nq)}; N_H | \hat{X}) \end{aligned} \quad \text{(A12)}$$

where $X_B^{(nq)}$ denotes the Nyquist samples process of the bandlimited process X_B, and the conditional Pinsker rate is defined as in (19), conditioned on the entire process $\hat{X}^{(nq)}$. The rate $\tilde{I}$ is defined, for an arbitrary process (variable) U, as

$$\tilde{I}(X_B^{(nq)}; U) = 2B \cdot \limsup_{N\to\infty} \frac{1}{n} I(\boldsymbol{X}_B^{(nq)}; U) \quad \text{(A13)}$$

where $\boldsymbol{X}_B^{(nq)}$ denotes n-tuples of $X_B^{(nq)}$ (see [16, p. 76], and [9, p. 141]).

The chain of equalities in (A12) holds as follows: (a) follows from (A10); (b) follows from the data-processing theorem and the fact that the sample function $x_B^{(nq)}$ has a one-to-one relation with x_q (as in (A10)-(c)); (c) follows from the assumption that X is nondegenerate, and thus $X_B^{(nq)}$ has the "finite-gap information property" (20) and $\overline{I}^{(g)}$ can be replaced with $\tilde{I}$; for (d) and (e) we first observe that there is a linear deterministic one-to-one relation between the sample function $\tilde{x} = \hat{x} + n_H$ and the pair of sample functions $\{\hat{x}, n_H\}$, since they occupy nonoverlapping frequency bands; then, we use the fact that the second argument in $\tilde{I}(\cdot; \cdot)$ is considered as a random *variable* rather than a process, and so it may be manipulated according to the basic properties of mutual information, i.e., it is invariant under an invertible transformation and the chain rule; finally, (f) follows similarly to (c), since X is nondegenerate.

We continue with another chain of equalities

$$\begin{aligned}\tilde{I}(N_H^{(nq)};\, N_B) &\overset{(a)}{=} \tilde{I}(N_H^{(nq)};\, N_B,\, X_B)\\ &\overset{(b)}{=} \tilde{I}(N_H^{(nq)};\, \hat{X},\, X_B)\\ &\overset{(c)}{=} \tilde{I}(N_H^{(nq)};\, \hat{X}) + \tilde{I}(N_H^{(nq)};\, X_B|\hat{X}) \quad \text{(A14)}\end{aligned}$$

where we use the fact that X_B is independent of both N_B and N_H, and apply the same technique as in equalities (d) and (e) of (A12). Now, in Subsection B where we showed that $N_H^{(nq)}$ is smooth, and thus (A14) also holds for the corresponding Pinsker rates. Combining with (A12) we get

$$R_Q = \overline{I}^{(g)}(X_B^{(nq)};\, \hat{X}) + \overline{I}^{(g)}(N_H^{(nq)};\, N_B) - \overline{I}^{(g)}(N_H^{(nq)};\, \hat{X}). \quad \text{(A15)}$$

The desired result, (24), is obtained by replacing the Nyquist sampled processes with their equivalent bandlimited processes, and changing X_B to X, as in (A10)-(c).

D. Low-Distortion Behavior of $r(D)$

The term $r(D)$ which appears, e.g., in Theorem 4 is analogous to the resolution measure defined in [22]. In [14] it was shown that $r(D)$ vanishes, as $D \to 0$, for vector sources with a finite differential entropy. In this subsection we show a similar result for discrete-time processes with a finite entropy rate, and in some cases we even characterize the convergence rate.

Let us first recall the following lemma due to [13]:

Lemma 2: Let $X = X_1, X_2, \cdots$ and $N = N_1, N_2, \cdots$ be stationary processes with finite powers, $EX_i^2 < \infty$, $EN_i^2 < \infty$, and assume that

$$\overline{h}(X) \triangleq \lim_{n\to\infty} \frac{1}{n} h(X_1 \cdots X_n)$$

exists and is finite, then

$$\lim_{\alpha\to 0} \overline{h}(X + \alpha \cdot N) = \overline{h}(X) \quad \text{(A16)}$$

where $X + \alpha N = X_1 + \alpha N_1,\, X_2 + \alpha N_2, \cdots$.

We are now ready to investigate $r(D) = \overline{I}^{(g)}(N_B;\, X_B + N_B)$ as we vary D, the MSE distortion level of the coding scheme. It is assumed that $D = E\{N_B^2(t)\}$ is varied by scaling a lattice quantizer which has a given structure, at a fixed sampling rate F_s. We claim the following.

Lemma 3: For smooth sources, $r(D) \to 0$ as $D \to 0$.

Proof: Since the Nyquist sampled process of N_B is smooth, we may write, as in (A6) above

$$r(D) = \lim_{T\to\infty} \frac{1}{T}[h(\boldsymbol{X}_B^{(T)} + \boldsymbol{N}_B^{(T)}) - h(\boldsymbol{X}_B^{(T)})]. \quad \text{(A17)}$$

Now, since the distortion is varied by scaling Q_K, we may substitute $\boldsymbol{N}_B^{(T)} = \sqrt{D} \cdot \tilde{\boldsymbol{N}}_B^{(T)}$, where $\tilde{N}_B^{(T)}$ is the Nyquist samples vector of the quantization noise in the case $D = 1$, i.e., when $\epsilon = F_s/2B$. Furthermore, since the source is smooth

$$\lim_{T\to\infty} \frac{1}{T} h(\boldsymbol{X}_B^{(T)}) = 2B \cdot \overline{h}_x > -\infty$$

and $EX_B^2 = \sigma_x^2 < \infty$. Hence, Lemma 2 implies

$$\lim_{D\to 0}\lim_{T\to\infty} \frac{1}{T} h(\boldsymbol{X}_B^{(T)} + \sqrt{D}\cdot \tilde{\boldsymbol{N}}_B^{(T)}) = \lim_{T\to\infty} \frac{1}{T} h(\boldsymbol{X}_B^{(T)})$$

which proves this lemma. ■

It turns out, as shown in [13], that Lemma 3 above holds not only for the MSE distortion measure, but rather for a larger class of distortion measures.

In some cases, the low-distortion behavior of $r(D)$ may be expressed more explicitly. We consider here two such cases: the case of a Gaussian source, and the case of a Gaussian noise which is associated either with large lattice dimension (see Section VI) or with high sampling rate. In these two examples $r(D) = O(D)$ for small D.

If the source is Gaussian, denoted X^*, we may upper bound $r(D)$ by

$$\begin{aligned}r(D) &= \overline{I}^{(g)}(N_B;\, N_B + X^*) \le \overline{I}^{(g)}(N_B^*;\, N_B^* + X^*)\\ &= \int_0^B \log\left(1 + \frac{D/B}{S_x(f)}\right) df\\ &= D \cdot \left(\frac{\log e}{B}\int_0^B \frac{df}{S_x(f)}\right) + O(D^2) \quad \text{(A18)}\end{aligned}$$

where N_B^* denotes Gaussian quantization noise, with the same spectrum as N_B, and the last equality holds if $S_x(f) \ge s > 0$ for some s, at all frequencies. For a Gaussian source with flat spectrum, (A18) reduces to

$$r(D) \le D \cdot \left(\frac{B \log e}{\sigma_x^2}\right) + O(D^2).$$

If the noise is Gaussian, we assume that $h(\boldsymbol{X}_B^{(T)} + \sqrt{\alpha} \cdot \tilde{\boldsymbol{N}}_B^{*(T)})$ is twice differentiable with respect to α in the neighborhood of $\alpha = 0$, and we use the well known De-Bruijn's identity

$$\frac{d}{d\alpha} h(\boldsymbol{X}_B^{(T)} + \sqrt{\alpha}\cdot \tilde{\boldsymbol{N}}_B^{*(T)}) = \frac{\log e}{2} J(\boldsymbol{X}_B^{(T)})$$

where $J(\boldsymbol{X}_B^{(T)})$ is the *Fisher information* of $\boldsymbol{X}_B^{(T)}$, to obtain

$$\begin{aligned}r(D) &= \overline{I}^{(g)}(N_B^*;\, N_B^* + X_B)\\ &= \lim_{T\to\infty} \frac{1}{T}[h(\boldsymbol{X}_B^{(T)} + \sqrt{D}\cdot \tilde{\boldsymbol{N}}_B^{(T)}) - h(\boldsymbol{X}_B^{(T)})]\\ &= D \cdot \frac{\log e}{2} \cdot \lim_{T\to\infty} \frac{1}{T} J(\boldsymbol{X}_B^{(T)}) + O(D^2). \quad \text{(A19)}\end{aligned}$$

If the source is Gaussian with flat spectrum

$$\frac{1}{T} J(\boldsymbol{X}_B^{(T)}) = \frac{n}{T}\cdot\frac{1}{\sigma_x^2} = \frac{2B}{\sigma_x^2}$$

and (A19) is consistent with the previous case.

E. Proof of Lemma 1

As in the proof of Theorem 3, $X \to X_B \to \hat{X}$ forms a Markov chain, implying $\overline{I}^{(g)}(X;\, \hat{X}) = \overline{I}^{(g)}(X_B;\, X_B + N_B)$.

By a similar decomposition to that in (A17) and similarly to the decomposition in (A8) we obtain

$$\overline{I}^{(g)}(X_B;\, X_B+N_B) = r(D) + 2B\cdot \overline{h}_x + \overline{\mathcal{D}}^{(g)}(N_B;\, N_B^*) - 2B\cdot \tfrac{1}{2}\log(2\pi e D) \quad \text{(A20)}$$

and substitute $\overline{h}_x = \frac{1}{2}\log(2\pi e P_x)$ to complete the proof of the lemma.

F. Proof of Theorem 5

Let $U = \{U(t)\}$ be a process, jointly stationary with the source X and independent of the noise N, and let $U_q = \{U_q[n]\}$ be obtained from U by low-pass filtering to bandwidth B and sampling at a rate F_s (in the same way X_q is obtained from X). It follows from the proof of [22, Theorem 2] that, for any such U and any block length n

$$I(\boldsymbol{X}_q;\, \boldsymbol{X}_q+\boldsymbol{N}_q) \le I(\boldsymbol{X}_q;\, \boldsymbol{U}_q) + I(\boldsymbol{X}_q-\boldsymbol{U}_q;\, \boldsymbol{X}_q-\boldsymbol{U}_q+\boldsymbol{N}_q) \quad \text{(A21)}$$

where $\boldsymbol{X}_1$, $\boldsymbol{N}_q$, and $\boldsymbol{U}_q$ are n-vectors of the corresponding processes. Dividing by n and taking the limit, we get the same inequality for the information rates, i.e.

$$\overline{I}(X_q;\, X_q+N_q) - \overline{I}(X_q;\, U_q) \le \overline{I}(X_q-U_q;\, X_q-U_q+N_q) \quad \text{(A22)}$$

provided that $\overline{I}(X_q;\, U_q)$ exists. By (13) and (23) the leftmost term of (A22) satisfies

$$F_s\cdot \overline{I}(X_q;\, \hat{X}_q) = R_Q = \overline{I}^{(g)}(X;\, \tilde{X})$$

and similarly (by the smoothness of N_q) the rightmost term satisfies

$$F_s\cdot \overline{I}(X_q-U_q;\, \widehat{X_q-U_q}) = \overline{I}^{(g)}(X-U;\, \widetilde{X-U}).$$

Now, suppose in addition that $E(X-U)^2 \le D$, implying $E(X_q-U_q)^2 \le D$. Then we may write

$$R_Q(D) - R(D) = F_s\cdot\left(\overline{I}(X_q;\, \hat{X}_q) - \inf_U \overline{I}(X_q;\, U_q)\right) \le \sup_{X,\,U} \overline{I}^{(g)}(X-U;\, \widetilde{X-U}) = C \quad \text{(A23)}$$

where the infimum and supremum are under the constraint that $E(X-U)^2 \le D$, and

$$R(D) = \inf_U \overline{I}^{(g)}(X;\, U) = F_s\cdot \inf_U \overline{I}(X_q;\, U_q)$$

by (26) and by [9, Theorem 10.6.1]. Note that we actually proved a somewhat stronger claim, that for any D and ϵ, $R_Q(\epsilon) - R(D) \le C\ (SNR = D/\epsilon)$.

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